Lipschitz fcts & $W^{1,\infty}$

Theorem 4 (Characterization of $W^{1,\infty}$)

Let Ω be open and bdd, w/ $\partial\Omega$ of class C^1. Then $u:\Omega \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(\Omega)$.

Proof

Let u be arbitrary and consider the extension $Eu = \bar{u}$, then (from Thm 1, §5.4) $Eu = \bar{u}$ a.e. in Ω & Eu has compact support within $\mathbb{R}^n$

Now assume that $u \in W^{1,\infty}(\Omega) \Rightarrow \bar{u} \in W^{1,\infty}(\mathbb{R}^n)$.
Then $\bar{u}^\varepsilon := \eta_\varepsilon * \bar{u}$, where η_ε is the standard mollifier, is smooth and satisfies

$$\begin{cases} \bar{u}^\varepsilon \longrightarrow \bar{u} \text{ uniformly as } \varepsilon \to 0, \\ \|D\bar{u}^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \|D\bar{u}\|_{L^\infty(\mathbb{R}^n)} \end{cases}$$

> **Recall**
>
> i) $\eta \in C^\infty(\mathbb{R}^n)$ is defined by
>
> $$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$
>
> where $C > 0$ s.t. $\int_{\mathbb{R}^n} \eta \, dx = 1$.
>
> ii) For each $\varepsilon > 0$, set $\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$, we call η_ε the **standard mollifier.** The fcts η_ε are C^∞ & satisfy
>
> $$\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1, \quad spt(\eta_\varepsilon) \subset B(0,\varepsilon).$$

Let $x, y \in \mathbb{R}^n$ be arbitrary, s.t. $x \neq y$, then we have

$$\bar{u}^\varepsilon(x) - \bar{u}^\varepsilon(y) = \int_0^1 \frac{d}{dt} \bar{u}(tx + (1-t)y)\,dt$$
$$= \int_0^1 D\bar{u}^\varepsilon(tx + (1-t)y)\,dt \cdot (x-y)$$

which implies

$$|\bar{u}^\varepsilon(x) - \bar{u}^\varepsilon(y)| \leq \|D\bar{u}^\varepsilon\|_{L^\infty(\mathbb{R}^n)} |x-y| \leq \|D\bar{u}\|_{L^\infty(\mathbb{R}^n)} |x-y|.$$

Let $\varepsilon \longrightarrow 0$ then

$$|\bar{u}(x) - \bar{u}(y)| \leq \|D\bar{u}\|_{L^\infty(\mathbb{R}^n)} |x-y|$$

for all $x, y \in \mathbb{R}^n$. Restrict x, y to Ω & we have $u : \Omega \longrightarrow \mathbb{R}$ is Lipschitz continuous. ✓✓

Assume $u : \Omega \longrightarrow \mathbb{R}$ is Lipschitz continuous and therefore

$$\|D_i^{-h} u\|_{L^\infty(\Omega)} \leq \mathrm{Lip}(\Omega),$$

for each fixed $i = 1, \dots, n$, and thus there exists a fct $v_i \in L^\infty(\Omega)$ and a subseq. $h_k \to 0$ s.t.

$$D_i^{-h_k} u \longrightarrow v_i \qquad \text{weakly in } L^2_{loc}(\Omega).$$

Let $\phi \in C_c^\infty(\Omega)$ and consequently

$$\int_{\mathbb{R}^n} u \phi_{x_i} dx = \lim_{h_k \to 0} \int_{\mathbb{R}^n} u (D_i^{h_k} \phi) dx$$
$$= -\lim_{h_k \to 0} \int_{\mathbb{R}^n} (D_i^{-h_k} u) \phi \, dx$$
$$= -\int_{\mathbb{R}^n} v_i \phi \, dx$$

which holds for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and so $v_i = u_{x_i}$ (for $i = 1, \dots, n$). Since $u_{x_i} = v_i \in L^\infty_{loc}(\Omega)$ we have $u \in W^{1,\infty}(\Omega)$

□

Differentiability a.e.

Definition (differentiable).

A fct $u: \Omega \to \mathbb{R}$ is **differentiable** at $x \in \Omega$ if there exists $a \in \mathbb{R}^n$ such that

$$u(y) = u(x) + a \cdot (y-x) + o(|y-x|) \qquad \text{as} \quad y \to x$$

i.e.

$$\lim_{y \to x} \frac{|u(y) - u(x) - a \cdot (y-x)|}{|y-x|} = 0.$$

If a exists it is unique and we write $a = Du(x)$, which is called the **gradient** of u.

Theorem (Differentiability almost everywhere)

Assume $u \in W^{1,p}_{loc}(\Omega)$ for some $n < p \le \infty$. Then u is differentiable a.e. in Ω, and it's gradient equals it's weak gradient a.e.

Proof

Assume $n < p < \infty$. Note that $W^{1,\infty}(\Omega) \subset W^{1,p}(\Omega)$ and therefore everything that follows holds for $p = \infty$ (but requires different notation). A variant of the proof of Morrey's inequality provides

Morrey's is kosher for $n < p \le \infty$, but this variant (somewhat obviously) only holds for $n < p < \infty$.

$$|v(y) - v(x)| \le C r^{1-n/p} \left(\int_{B(x,2r)} |Dv(z)|^p dz \right)^{1/p} \qquad (1)$$

for all $y \in B(x,r)$, & all C^1 fcts v and therefore (by approximation) any $v \in W^{1,p}$. C depends only on p & n.

Let $u \in W^{1,p}_{loc}(\Omega)$. Now for almost every $x \in \Omega$, Lebesgue Differentiation Theorem (§E.4) (requires local summability) implies

$$⨍_{B(x,r)} |Du(x) - Du(z)|^p dz \to 0 \qquad \text{as} \quad r \to 0.$$

Where Du is the weak derivative of u!

Fix an arbitrary $x \in \Omega$ and define

$$v(y) := u(y) - u(x) - Du(x)\cdot(y-x)$$

then note that $v(x) = 0$. Estimate ① is now

$$\begin{aligned}
|u(y)-u(x)-Du(x)\cdot(y-x)| &\le Cr^{1-n/p}\left(\int_{B(x,2r)} |D[u(z)-u(x)-Du(x)\cdot(z-x)]|^p dz\right)^{1/p} \\
&= Cr^{1-n/p}\left(\int_{B(x,2r)} |Du(z)-Du(x)|^p dz\right)^{1/p} \\
&= Cr^{1-n/p}\left(Cr^n ⨍_{B(x,2r)} |Du(z)-Du(x)|^p dz\right)^{1/p} \\
&\le Cr\left(⨍_{B(x,2r)} |Du(z)-Du(x)|^p dz\right)^{1/p} \\
&= o(r)
\end{aligned}$$

Thanks to Lebesgue Differentiation Theorem.

as $r \to 0$. Note that as $r \to 0$, $y \to x$ and therefore

$$|u(y)-u(x)-Du(x)\cdot(y-x)| = o(|y-x|) \quad \text{as } y \to x$$

which implies u is differentiable at a.e. $x \in \Omega$ and by uniqueness it's gradient equals it's weak gradient.

Theorem 6 (Rademacher's Thm)

Let u be locally Lipschitz continuous in Ω. Then u is differentiable almost everywhere in Ω.

Theorems 3,4,5 "mostly" imply this almost directly.

Notation

$\hat{u}$ is the Fourier transform of u, & $\check{u}$ is the inverse Fourier transform.

Fourier Methods

Recall, $H^k = W^{k,2}$, also it is Hilbert!

Theorem (Characterization of H^k by Fourier transform)

Let k be a nonnegative integer.

i) a fct $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if

$$(1+|y|^k)\hat{u} \in L^2(\mathbb{R}^n).$$

ii) In addition, there exists a $C>0$ s.t.

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \le \|(1+|y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{H^k(\mathbb{R}^n)}$$

for each $u \in H^k(\mathbb{R}^n)$.

Properties we will need:

1) if $u \in L^2(\mathbb{R}^n)$ then $\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)}$

2) $\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$

3) $\int_{\mathbb{R}^n} u\bar{v}\, dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}\, dy$

Proof

i) Assume first $u \in H^k(\mathbb{R}^n)$. Then for each multiindex $|\alpha| \le k$, we have $D^\alpha u \in L^2(\mathbb{R}^n)$. Now if $u \in C^k$ has compact spt, we have

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$$

according to Thm 2 in §4.3.1.

Approximating by smooth functions we deduce

$$\widehat{D^\alpha u} = (iy)^\alpha \hat{u}$$

for all $u \in H^k(\mathbb{R}^n)$ which implies $(iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for each $|\alpha| \le K$. Note that since $D^\alpha u \in L^2(\mathbb{R}^n)$ we have $\|\widehat{D^\alpha u}\|_{L^2(\mathbb{R}^n)} = \|D^\alpha u\|_{L^2(\mathbb{R}^n)}$. Then

$$\begin{aligned} \int |D^k u|^2 dx &= \int \sum_{|\alpha|=k} |D^\alpha u|^2 dx \\ &= \int \sum_{|\alpha|=k} |\widehat{D^\alpha u}|^2 dy \\ &= \int \sum_{|\alpha|=k} |(iy)^\alpha \hat{u}|^2 dy \\ &= \int \sum_{|\alpha|=k} |y|^{2\alpha} |\hat{u}|^2 dy \\ &\ge C \int |y|^{2k} |\hat{u}|^2 dy \end{aligned}$$

picking $\alpha = k\vec{e}_1$

Therefore,

$$\int |y|^{2k} |\hat{u}|^2 \le C \int |D^k u|^2$$

$a,b>0$ then

$$\begin{aligned} (a+b)^2 &\le (2\max\{a,b\})^2 \\ &= 2^2 (\max\{a,b\})^2 \\ &\le 2^2 \left[(\max\{a,b\})^2 + (\min\{a,b\})^2\right] \\ &= 2^2(a^2+b^2). \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^n} (1+|y|^k)^2 |\hat{u}|^2 dy &\le C \int_{\mathbb{R}^n} (1+|y|^{2k}) |\hat{u}|^2 dy \\ &\le C\left(\|\hat{u}\|_{L^2}^2 + C_1 \|D^k u\|_{L^2}^2\right) \\ &\le C \|u\|_{H^k}^2 \end{aligned}$$

$$\Rightarrow (1+|y|^k)\hat{u} \in L^2 \mathbb{R}^n \qquad \checkmark\checkmark$$

Now assume $(1+|y|^k)\hat{u} \in L^2(\mathbb{R}^n)$ & $|\alpha| \le k$. Then

$$\|(iy)^\alpha \hat{u}\|_{L^2(\mathbb{R}^n)}^2 \le \int_{\mathbb{R}^n} |y|^{2|\alpha|} |\hat{u}|^2 dy \le C \|(1+|y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)}^2.$$

$$\Rightarrow (iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n).$$

Define

$$u_\alpha := \left[(iy)^\alpha \hat{u}\right]^\vee.$$

Then for all $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (D^\alpha \phi)\bar{u}\,dx = \int_{\mathbb{R}^n} (\widehat{D^\alpha \phi})\bar{\hat{u}}\,dy$$
$$= \int_{\mathbb{R}^n} (iy)^\alpha \hat{\phi}\,\bar{\hat{u}}\,dy$$
$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \hat{\phi}\,\overline{(iy)^\alpha \hat{u}}\,dy$$
$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi\,\bar{u}_\alpha\,dy$$

$\Rightarrow u_\alpha = D^\alpha u$ in the weak sense and since $u^\alpha = (iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ this implies $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le k$ and we conclude that $u \in H^k(\mathbb{R}^n)$.

ii) follows immediately

$\square$

Definition

Assume $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$. Then $u \in H^s(\mathbb{R}^n)$ if $(1+|y|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For nonintegers s we set

$$\|u\|_{H^s(\mathbb{R}^n)} := \underbrace{\|(1+|y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}}_{\text{showed that this an equivalent norm to } L^2(\mathbb{R}^n)}.$$